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An Essay and Review of the Book: *Self Similar Processes*. Paul Embrechts and Makoto Maejima, Princeton University Press, 2003

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A self-similar processes is a stochastic process $\{X(t), t \geqslant 0\}$ whose finite-dimensional distributions have scaling properties, namely for any constant $a > 0$, the process

$$\{X(at), t \geqslant 0\} \tag{1}$$

has the same-finite dimensional distributions as

$$\{a^H X(t), t \geqslant 0\}. \tag{2}$$

This means that for any $t_1,..., t_n$ and any $x_1,..., x_n$, one has

$$\mathbb{P}\{X(at_1) \leqslant x_1,..., X(at_n) \leqslant x_n\} = \mathbb{P}\{a^H X(t_1) \leqslant x_1,..., a^H X(t_n) \leqslant x_n\}.$$

The exponent H has various names. It is called the "exponent" or "index of self similarity," the "Hurst exponent" or sometimes the "fractal exponent." The term "fractal exponent" should be avoided because it is not the paths of $X(t)$ that are self-similar but only the distribution or law of the process. Consider, for example, Brownian motion $\{B(t), t \geqslant 0\}$ whose covariance function is

$$\langle B(s), B(t) \rangle = \sigma^2 \min(s, t),$$

where $\sigma^2 = \langle B^2(1) \rangle$ and where $\langle\ \rangle$ or $\langle\ ,\ \rangle$ denotes the mean value (mathematical expectation). Since Brownian motion has mean zero, $\langle B(t) \rangle = 0$, and is a Gaussian process, its finite-dimensional distributions are determined by the covariance function, and since, for any $a \geqslant 0$,

$$\langle B(as), B(at) \rangle = a\sigma^2 \min(s, t) = \langle a^{1/2} B(s), a^{1/2} B(t) \rangle,$$

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one concludes that Brownian motion is self-similar with $H = 1/2$. Observe that we have not analyzed the path $\{B(\omega, t), t \geqslant 0\}$ for a realization ω of the process. The path $\{B(\omega, t), t \geqslant 0\}$ does not have scaling or self-similarity property. If it did, then knowing the value of $B(\omega, t_0)$ at any time t_0, one would be able to derive the values of $B(\omega, at_0)$ at any subsequent time $at_0 > t_0$ with $a > 1$. The self-similarity exponent H does influence the path properties, however. For example, the paths of Brownian motion are Hölder-continuous of order $H = 1/2$, that is

$$|B(\omega, s) - B(\omega, t)| \leqslant C(\omega)\, |s-t|^{\gamma},$$

for any $\gamma < 1/2$.

There are a lot of self-similar processes, in fact as many as stationary processes. Indeed, one can always transform a self-similar process into a stationary process and vice-versa. Thus one often focuses on *H-sssi processes*, that is, self-similar processes that have also *stationary increments*. Brownian motion, for example, is $1/2$-sssi. The H-sssi processes are important in practice because if $\{X(t), t \geqslant 0\}$ is such a process, then its increments

$$Y_i = X(i+1) - X(i), \qquad i \geqslant 0$$

form a stationary sequence, which is often used in modeling real-life phenomena. If X is Brownian motion, then the Y_i's are independent and identically distributed Gaussian random variables. To obtain less trivial models, one has to focus on other H-sssi processes X. If we want to maintain the Gaussian nature of the process, only the covariance function has to change. Gaussian H-sssi processes are known as *fractional Brownian motion*, or FBM in short. They are denoted $\{B_H(t), t \geqslant 0\}$, where $0 < H < 1$. The covariance function of FBM is

$$\langle B_H(s), B_H(t) \rangle = \frac{\sigma^2}{2} \{|s|^{2H} + |t|^{2H} - |s-t|^{2H}\}, \tag{3}$$

where $\sigma^2 = \langle B_H^2(1) \rangle$. The corresponding increments $Y_i = B_H(i+1) - B_H(i)$, $i \geqslant 0$ are called *fractional Gaussian noise* (FGN) and they display long-range dependence when $1/2 < H < 1$:

$$\langle Y_i Y_{i+k} \rangle \sim c k^{2H-2} \qquad \text{as} \quad k \to \infty, \tag{4}$$

where $\sim$ means "asymptotic to" (the ratio of the left and right hand sides tends to 1) and where $c > 0$ is a constant. They thus decay slowly at large lags k and, in fact, the spectral mass at frequency zero is infinite:

$$\sum_{k=-\infty}^{\infty} \langle Y_i, Y_{i+k} \rangle = \langle Y_i^2(0) \rangle + 2 \sum_{k=1}^{\infty} \langle Y_i, Y_{i+k} \rangle = \infty, \tag{5}$$

and the spectral density approaches ∞ as the frequency tends to zero, like a power function, a phenomena, sometimes called "1/f noise."

Brownian motion does not display long-range dependence. It is FBM with $H = 1/2$. FBM with $H \neq 1/2$ was invented by Kolmogorov[3] and rendered popular by Mandelbrot in many articles, in particular Mandelbrot and Van Ness,[4] where the connection between fractional Brownian motion and fractional integration was made. In fact, one can represent FBM as a stochastic integral

$$B_H(t) = \int_{-\infty}^{\infty} f(t, s)\, dB(s) \tag{6}$$

by using a non-random integrand f, suitably chosen (it resembles a power function). The integration in (6) is with respect to Brownian motion, where $dB(s)$ can be interpreted as Gaussian white noise. The representation (6) is useful because it sheds light on the structure of FBM, and, more importantly, because it opens the door to extensions to the non-Gaussian world.

The extension to the non-Gaussian world can be done by considering either finite variance or infinite variance processes. The idea, in both cases, is to extend the integral representation (6) in one case, by considering a multiple integral, that is

$$B_H(t) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(t, s_1,..., s_p)\, dB(s_1) \cdots dB(s_p) \tag{7}$$

with a suitably chosen f. Such an extension is related to the so-called "Wiener Chaos." Since an H-sssi process of the type (7) has finite variance, its covariance is still (3) but its higher moments can be rather complicated, and in fact, are best described by "diagram formulas," akin to Feynman diagrams. The process (7) is clearly not Gaussian when $p > 1$ because $dB(s_1) \cdots dB(s_p)$ involves intuitively the product of p Gaussian variables.

Another way to extend H-sssi to the non-Gaussian world is to replace the Gaussian white noise dB in the stochastic representation (6) of FBM by so-called "stable" white noise with infinite variance. Recall that if a random variable M_α has a "stable" distribution, then $\mathbb{P}\{|M_\alpha| > x\} \backsim cx^{-\alpha}$

as $x \to \infty$ where $0 < \alpha < 2$ and $c > 0$ is a constant. One says that M_α is stable or that M_α has heavy tails (M_α has infinite variance because $\alpha < 2$). One can also define a stable white noise $dM_\alpha(s)$ and hence consider the H-sssi process

$$X(t) = \int_{-\infty}^{\infty} f(t, s)\, dM_\alpha(s) \tag{8}$$

where f is suitably chosen. If the noise $dM_\alpha(s)$ has a stable distribution then the resulting process $X(t)$ has also a stable distribution and hence heavy tails. The monograph of Samorodnitsky and Taqqu[(6)] offers a systematic introduction to stable non-Gaussian random processes, including H-sssi ones.

The following two bibliographical articles, Taqqu[(8)] and Willinger *et al.*,[(11)] may be also be useful. They are a somewhat dated but they are annotated and point to a large body of literature covering both theory and applications.

The book by Embrechts and Maejima under review offers a nice introduction to self similar processes. It is short (111 pages), assumes notions of probability theory at an introductory graduate level and covers many topics associated with self-similar processes. It describes the general properties of H-sssi processes and, in particular, of fractional Brownian motion. It also includes a number of sections on limit theorems. Limit theorems are important because self-similarity is an idealized property, and in practice, many real-life phenomena are only approximately self-similar. They converge to self-similar processes when a parameter included in the model converges to infinity.

The fact that fractional Brownian motion is not a semi-martingale when $H \neq 1/2$ is established. This is of relevance to finance. A semi-martingale is roughly the sum of a process of bounded variation (for example a smooth continuous function) and a martingale, which I will define. The bounded variation process can be ignored when pricing a stock. The martingale component, however, is of great importance. Suppose $\{X(t), t \geqslant 0\}$ represents the value of the stock. It is a martingale, if the conditional expectation

$$\mathbb{E}\{X(t_2) - X(t_1) \mid X(s), 0 \leqslant s \leqslant t_1\} = 0, \qquad t_1 < t_2, \tag{9}$$

that is, if with knowledge of all information up to the present, the average gain increase in the future is 0. Thus, a martingale is a model for a fair game. Brownian motion, for example, is a martingale, and in fact, Brownian motion is ubiquitous in finance. When the price process is a semi-martingale then one cannot have arbitrage opportunities (possible gains with any risk of loss). Fractional Brownian motion with $H \neq 1/2$ is neither a martingale nor a semi-martingale. It can therefore not be used in finance as a model for changes of stock prices. It is the local behavior of the fractional Brownian motion paths and not the long-term dependence structure of the process that is responsible for this. There are slight modifications of fractional Brownian motion that make it a semi-martingale.

The authors also cover briefly stable processes and define the various types of H-sssi stable processes (such as moving averages, harmonizable, sub-Gaussian processes). In the stable case, the dependence structure cannot be described by covariances because these are infinite. There are partial descriptions of dependence, for example the "codifference" or the "covariation." These reduce to the covariance when the process is Gaussian.

The book includes also a discussion of self-similar process whose increments are independent but not necessarily stationary. The marginal distribution of $X(t)$ for a fixed t can be characterized. It belongs to the so-called class of "selfdecomposable" distributions, defined in the book, which is a more general class than the class of stable distributions. Unfortunately, in contrast to the stable case, the finite-dimensional distributions are not, in general, selfdecomposable.

The authors also discuss the sample path properties of H-sssi processes (e.g., Hölder continuity). They include a chapter on how to simulate self-similar processes. A Gaussian sequence can be exactly simulated, for example, by using the Durbin–Levinson algorithm. There are various approximations for the simulation of fractional Brownian motion, some use the fast Fourier transforms, other wavelets. A short chapter is devoted to statistical estimation. It includes the historical "R/S statistic" (which is very biased) and describes some maximum likelihood methods for estimating the exponent H. The final chapter, called "Extensions," contains a brief discussion on "operator selfsimilar" processes which take values in $\mathbb{R}^d$ and where the constant a in (1) is replaced by a matrix. It also contains an introduction to "semi-selfsimilar processes" where a^H in (2) is replaced by some arbitrary function of a (see Sato[(7)]).

The book is written in a rigorous mathematical way with "definition," "theorem," etc. It contains a number of proofs, some of which are only sketches. While most of the results are given without proof, a reference to the original paper or to a book where the proof can be found is provided. This way of writing is useful for someone who wants to get a general overview of the subject but at the same time wants facts that are stated in a precise way.

The authors do not discuss in depth statistical issues related to the detection of self-similarity and the estimation of H. Nor do they cover practical applications, for example, to telecommunications, hydrology, or finance. This would not be possible in such a short book. The reader interested in these subjects can refer to Beran[1] which focuses on statistics, to an applied overview article on statistical techniques by Taqqu and Teverovsky,[10] to a review article on the "on-off" models in telecommunications by Taqqu[9] and to two recent (edited) monographs.

The first monograph, Rangarajan and Ding,[5] contains articles among others, by Mandelbrot on fractal sum of pulses, Stanley *et al.* on patterns and correlations in economic phenomena, Gorenflo and Mainardi on fractional diffusion processes, Silverberg and Verspagen on long-memory and economic growth, Ivanov on heartbeat dynamics.

The second monograph is by Doukhan *et al.*[2] and is divided in two parts. The first part contains an extensive introductory article on fractional Brownian motion and long-range dependence, and articles involving the probabilistic properties of self-similar processes including the diagrams formulas referred to above, fractional calculus, and articles on statistics (parametric, semiparametric, and nonparametric estimation of H). The second part is devoted to applications and methodology and contains articles on applications to data network traffic, finance, hydrology, turbulence. The methodology articles describe and compare many methods for simulating self-similar processes and for estimating H, including wavelet methods. These two monographs complement nicely the book by Embrechts and Maejima under review.

I would thus advise the reader interested in learning about self-similar processes to read first the introductory articles on fractional Brownian motion in these two monographs, then to turn to the book of Embrechts and Maejima for a more extensive overview of the subject and finally to go back to the monographs for more specialized articles.

The book of Embrechts and Maejima can be used as a text for a one-semester course. It is geared toward mathematicians but anyone who has had an introductory graduate course in probability can easily read it. It is nicely written and provides a quick and excellent overview of the subject.

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